Journal of Mathematical Analysis and Applications **237,** 405–412 (1999)
Article ID jmaa.1999.6422, available online at http://www.idealibrary.com on IDEAL®

Projection Methods for Monotone Variational Inequalities

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Submitted by William F. Ames

Received January 19, 1999

In this paper, we study some new iterative methods for solving monotone variational inequalities by using the updating technique of the solution. It is shown that the convergence of the new methods requires the monotonicity and pseudomonotonicity of the operator. The new methods are very versatile and are easy to implement. The techniques include the splitting and extragradient methods as special cases.

Key Words: variational inequalities; projection method; splitting method; convergence criteria.

1. INTRODUCTION

Variational inequalities are being used to interpret the basic principles of mathematical and physical sciences in a simple and elegant way. Variational inequalities provide us with a general and a unified framework to study a wide class of problems arising in pure and applied sciences (cf. [1–19] and the references therein). There are a substantial number of numerical methods including the projection method and its variant forms,

0022-247X/99 $30.00

Wiener–Hopf equations, auxiliary principle technique, and decomposition methods. One of the most important methods is the projection method, which is mainly due to Sibony [16]. This method has the drawback that it requires the underlying operator T to be strongly monotone and Lipschitz continuous. This requirement limits the choice of the applications to a large number of problems. A number of modifications have been considered for removing the requirement of strong monotonicity for convergence (cf. [2, 17]). The extragradient method [2, 11, 17] overcomes this difficulty by the technique of updating the solution. This method is easy to implement, uses little storage, and can readily exploit any sparsity or separable structure in the operator or in the space. For applications of the extragradient methods one is referred to [2, 19].

In this paper, we propose a new class of iterative methods that are as versatile and as capable of exploiting problem structure as the extragradient method and, yet, are even simpler than the latter and have a scaling feature that is absent in the extragradient method. Our analysis is also similar in spirit to those of the methods of He [7, 8], and Noor [12, 13], but our methods differ from their approach. The convergence of these new methods requires the monotonicity and pseudomonotonicity of the underlying operator. Our method of convergence is very simple and is easy to implement. In Section 2, we formulate the problem and review the relevant concepts. The main results are discussed in Section 3.

2. PRELIMINARIES

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot , \cdot \rangle$ and $\| \ \|$ respectively. Let K be a closed convex set in H and let $T\colon K \to H$ be a nonlinear operator.

Consider the problem of finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \text{for all } v \in K. \tag{2.1}$$

Problem (2.1) is called the *variational inequality problem*, which was considered and studied by Stampacchia [18] in 1964. For applications, numerical methods, sensitivity analysis, and formulations of the variational inequalities one may see [1–19].

If $K^* = \{u \in H : \langle u, v \rangle \geq 0, \text{ for all } v \in K\}$ is a polar convex cone of the convex cone K in H, then problem (2.1) is equivalent to finding $u \in K$ such that

$$Tu \in K^* \quad \text{and} \quad \langle Tu, u \rangle = 0, \tag{2.2}$$

which is called *the complementarity problem*.

For applications and numerical methods of complementarity problems, see [4, 9, 10, 12, 13, 15].

We need the following concepts and results, which play an essential role in our study.

LEMMA 2.1. *For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.3}$$

if and only if

$$u = P_K z,$$

where P_K is the projection of H onto the closed convex set K. Furthermore, P_K is nonexpansive.

DEFINITION. An operator $T: H \to H$ is said to be *monotone* if

$$\langle Tu - Tv, u - v \rangle \geq 0, \qquad \text{for all } u, v \in H.$$

and pseudomonotone if

$$\langle Tu, v - u \rangle \geq 0 \text{ implies}$$

$$\langle Tv, v - u \rangle \geq 0, \text{ for all } u, v \in H.$$

Note that monotonicity implies pseudomonotonicity, but the converse is not true, see [5].

3. MAIN RESULTS

In this section, we suggest and analyze some new iterative methods for solving the monotone variational inequalities (2.1). For this purpose, we need the following result, which can be proved by invoking Lemma 2.1.

LEMMA 3.1. *The function $u \in K$ is a solution of the variational inequality* (2.1) *if and only if $u \in K$ satisfies the relation*

$$u = P_K[u - \rho Tu], \tag{3.1}$$

where P_K is the projection of H onto K and $\rho > 0$ is a constant.

From Lemma 3.1, we see that problem (2.1) is equivalent to the fixed point problem (3.1). This alternative formulation is very useful from both theoretical and practical points of view and is used to suggest and and analyze the following iterative method.

ALGORITHM 3.1. For a given $u_0 \in K$, compute u_{n+1} by the iterative method

$$u_{n+1} = P_K[u_n - \rho Tu_n], \qquad n = 0, 1, 2, \dots .$$

For the convergence analysis of Algorithm 3.1, see Sibony [16], if the operator T is strongly monotone and Lipschitz continuous. We remark that the strong monotonicity requirement for the convergence of this algorithm is a serious drawback. In order to overcome this difficulty, one can modify the projection method by updating the solution u in the following way. The relation (3.1) can be written as

$$u = P_K[u - \rho T P_K[u - \rho T u]], \tag{3.2}$$

which is another fixed point formulation. This fixed point formulation is used to suggest and analyze the following iterative method, which is known as the extragradient method (cf. [6, 7, 11, 13–15, 19]).

ALGORITHM 3.2. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho T P_K[u_n - \rho T u_n]], \qquad n = 0, 1, 2, \dots .$$

For the convergence analysis of Algorithm 3.2, see [2]. It is well known [2, 19] that the extragradient method converges for the Lipschitz continuous operator T. We remark that the Lipschitz continuity of the monotone operator T still limits the applications of the extragradient method. Fortunately, this difficulty can be avoided and this is the main motivation of the paper.

We now define the residue vector $R(u)$ by the relation

$$R(u) = u - P_K[u - \rho T P_K[u - \rho T u]]. \tag{3.3}$$

From Lemma 3.1, it is clear that $u \in K$ is a solution of the variational inequality (2.1) if and only if $u \in K$ is a zero of the equation

$$R(u) = 0. \tag{3.4}$$

For a positive step size $\gamma \in (0, 2)$, the equation (3.4) may be written as

$$\begin{aligned} u + \rho T u &= u + \rho T u - \gamma R(u) \\ &= u + \rho T u - \gamma\{u - P_K[u - \rho T P_K[u - \rho T u]]\}. \end{aligned}$$

This fixed point formulation allows us to suggest a new iterative method.

ALGORITHM 3.3. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n + \rho T u_n - \rho T u_{n+1} - \gamma R(u_n). \tag{3.5}$$

We note that for $\gamma = 1$, Algorithm 3.3 reduces to

ALGORITHM 3.4. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = (I + \rho T)^{-1}[P_K[u_n - \rho T P_K[u_n - \rho T u_n]] + \rho T u_n], \qquad n = 0, 1, 2, \dots,$$

which is a forward–backward splitting method.

It is worth mentioning that our implicit methods differ from the implicit methods of He [7] and Noor [12]. For appropriate and suitable choices of the operator T, convex set K, and Hilbert space H, one can obtain a number of new and known iterative methods for solving variational inequalities and complementarity problems as special cases.

For the convergence analysis of Algorithm 3.3, we need the following results, which are proved using the technique of He [7] and Noor [12].

THEOREM 3.1. *Let $\bar{u} \in K$ be a solution of* (2.1). *If T: $K \to H$ is a monotone operator, then*

$$\langle u - \bar{u} + \rho(Tu - T\bar{u}), R(u) \rangle \geq \| R(u) \|^2, \qquad \textit{for all } u \in K. \tag{3.6}$$

Proof. Let $\bar{u} \in K$ be a solution of (2.1). Then

$$\langle T\bar{u}, v - \bar{u} \rangle \geq 0, \qquad \text{for all } v \in K. \tag{3.7}$$

Taking $v = P_K[u - \rho T P_K[u - \rho Tu]]$ in (3.7), we obtain

$$\rho \langle T\bar{u}, P_K[u - \rho T P_K[u - \rho Tu]] - \bar{u} \rangle \geq 0. \tag{3.8}$$

Setting $z = u - \rho Tu$, $u = P_K[u - \rho T P_K[u - \rho Tu]]$, $v = \bar{u}$ in (2.3), and using (3.3), we obtain

$$\langle R(u) - \rho Tu, P_K[u - \rho T P_K[u - \rho Tu]] - \bar{u} \rangle \geq 0. \tag{3.9}$$

Adding (3.8) and (3.9), we have

$$\langle R(u) - \rho(Tu - T\bar{u}), P_K[u - \rho T P_K[u - \rho Tu]] - \bar{u} \rangle \geq 0,$$

which can be written as

$$\langle R(u) - \rho(Tu - T\bar{u}), u - \bar{u} - R(u) \rangle \geq 0, \qquad \text{using } (3.3). \tag{3.10}$$

Using the monotonicity of T, we obtain

$$\begin{aligned} &\langle u - u + \rho(Tu - T\bar{u}), R(u) \rangle \\ &\qquad \geq \| R(u) \|^2 + \rho \langle Tu - T\bar{u}, u - \bar{u} \rangle \geq \| R(u) \|^2, \end{aligned}$$

the required result.

THEOREM 3.2. *Let $\bar{u} \in K$ be a solution of* (2.1) *and let u_{n+1} be the approximate solution obtained from Algorithm* 3.3. *Then*

$$\begin{aligned} &\| u_{n+1} - \bar{u} + \rho(Tu_{n+1} - T\bar{u}) \|^2 \\ &\qquad \leq \| u_n - \bar{u} + \rho(Tu_n - T\bar{u}) \|^2 - \gamma(2 - \gamma) \| R(u_n) \|^2. \end{aligned} \tag{3.11}$$

Proof. From (3.5) and (3.6), we have

$$\begin{aligned}\|u_{n+1} - \bar{u} + \rho(Tu_{n+1} - T\bar{u})\|^2 &= \|u_n - \bar{u} + \rho(Tu_n - T\bar{u}) - \gamma R(u_n)\|^2 \\ &\leq \|u_n - \bar{u} + \rho(Tu_n - T\bar{u})\|^2 \\ &\quad - \gamma(2-\gamma)\|R(u_n)\|^2.\end{aligned}$$

THEOREM 3.3. *The approximate solution u_{n+1} obtained from Algorithm* 3.3 *converges to the solution $\bar{u} \in K$ of the variational inequality* (2.1).

Proof. Let $u \in K$ be a solution of (2.1). Then, from (3.11), it follows that the sequence $\{u_n\}$ is bounded. Furthermore, from (3.11), we have

$$\sum_{n=0}^{\infty} \gamma(2-\gamma)\|R(u_n)\|^2 \leq \|(u_0 - u) + \rho(Tu_0 - Tu)\|^2,$$

which implies that

$$\lim_{n\to\infty} R(u_n) = 0.$$

Let $\bar{u}$ be the cluster point of $\{u_n\}$. Since $\{u_n\}$ is a bounded sequence, it has a subsequence $\{u_{n_i}\}$, which converges to $\bar{u}$. By the continuity of the operators T and P_K, it follows that $R(u)$ is continuous and

$$R(\bar{u}) = \lim_{i\to\infty} R(u_{n_i}) = 0.$$

This shows that $\bar{u}$ is the solution of the variational inequality (2.1) and consequently

$$\|u_{n+1} - \bar{u} + \rho(u_{n+1} - T\bar{u})\|^2 \leq \|u_n - \bar{u} + \rho(Tu_n - T\bar{u})\|^2.$$

It follows that the sequence $\{u_n\}$ has exactly one cluster point and $\lim_{n\to\infty} u_n = \bar{u} \in K$ satisfies the variational inequality (2.1).

We now suggest another method, which does not require the computation of the solution implicitly. Convergence of this method requires only the pseudomonotonicity, which is weaker than monotonicity.

For a step size $\gamma \in (0, 2)$, the equation (3.4) may be written as

$$u = u - \gamma R(u) = u - \gamma\{u - P_K[u - \rho T P_K[u - \rho Tu]]\}.$$

This fixed point formulation enables us to suggest the following method.

ALGORITHM 3.5. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - \gamma R(u_n), \qquad n = 0, 1, 2, \ldots . \tag{3.12}$$

For $\gamma = 1$, this reduces to the extragradient Algorithm 3.2.

Following the technique of Theorem 3.1–3.3, one can easily study the convergence analysis of Algorithm 3.5. However, we include the main steps for the sake of completeness.

THEOREM 3.4. *Let* $\bar{u} \in K$ *be a solution of* (2.1) *and let* $T\colon K \to H$ *be a pseudomonotone operator. Then*

$$\langle u - \bar{u}, R(u) \rangle \geq \| R(u) \|^2, \qquad \text{for all } u \in K. \tag{3.13}$$

Proof. Since $T\colon K \to H$ is a pseudomonotone operator, for all $u, \bar{u} \in K$,

$$\langle T\bar{u}, P_K[u - \rho T P_K[u - \rho Tu]] - \bar{u} \rangle \geq 0$$

implies

$$\langle TP_K[u - \rho TP_K[u - \rho Tu]], P_K[u - \rho TP_K[u - \rho Tu]] - \bar{u} \rangle \geq 0,$$

from which, we have

$$\rho \langle Tu, u - \bar{u} - R(u) \rangle \geq 0. \tag{3.14}$$

Adding (3.10) and (3.14), we obtain

$$\langle R(u), u - \bar{u} - R(u) \rangle \geq 0.$$

which implies that

$$\langle u - \bar{u}, R(u) \rangle \geq \| R(u) \|^2,$$

the required result.

THEOREM 3.5. *The sequence* $\{u_n\}$ *generated by Algorithm* 3.5 *for variational inequalities* (2.1) *satisfies the inequality*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - \gamma(2 - \gamma) \| R(u_n) \|^2, \qquad \text{for all } \bar{u} \in K. \tag{3.15}$$

Proof. From (3.12) and (3.13), we obtain

$$\begin{aligned} \|u_{n+1} - \bar{u}\|^2 &= \| u_n - \bar{u} - \gamma R(u_n) \|^2 \\ &\leq \|u_n - \bar{u}\|^2 - \gamma(2 - \gamma) \| R(u_n) \|^2. \end{aligned}$$

Following the technique of Theorem 3.3 and by invoking Theorem 3.5, one can easily prove that the approximate solution u_{n+1} obtained from Algorithm 3.5 converges to the exact solution $\bar{u} \in K$, satisfying the variational inequality (2.1).

Remark 3.1. In this paper, we have suggested and analyzed a number of new iterative methods for solving the monotone variational inequalities and related complementarity problems. The convergence of these new methods requires the monotonicity and pseudomonotonicity of the underlying operator T, whereas the convergence of the extragradient methods requires the Lipschitz continuity of the monotone operator. This clearly shows that our new methods are more efficient than the extragradient methods. In fact, we have shown that the convergence of the extragradient method requires only the pseudomonotonicity.

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